Some models for longitudinal count data

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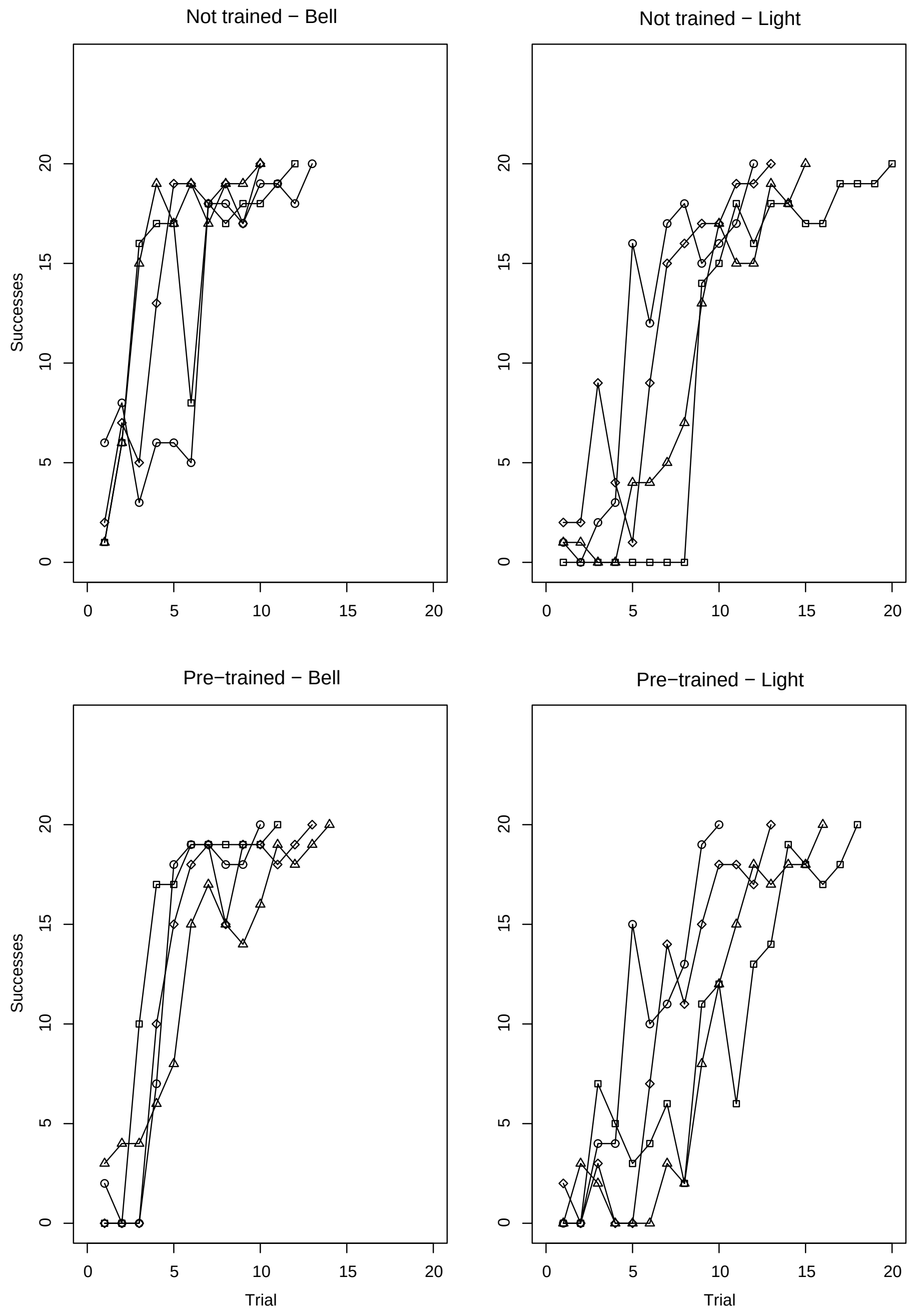
1 A learning experiment

16 laboratory animals were tested for learning in a 2×2 factorial experiment with training or not and light or bell stimulus. Each animal was allowed 20 attempts to complete a task in each of a series of trials (Aickin, 1983, pp. 238–240). Trials for an animal stopped when a perfect score was reached.

Not trained								Trained							
Light				Bell				Light				Bell			
6	1	2	1	1	0	2	1	2	0	0	3	0	0	2	0
8	6	7	6	0	0	2	1	0	0	0	4	0	0	0	3
3	16	5	15	2	0	9	0	0	10	0	4	4	7	3	2
6	17	13	19	3	0	4	0	7	17	10	6	4	5	0	0
6	17	19	17	16	0	1	4	18	17	15	8	15	3	0	0
5	8	19	19	12	0	9	4	19	19	18	15	10	4	7	0
18	18	18	17	17	0	15	5	19	19	19	17	11	6	14	3
18	17	19	19	18	0	16	7	18	19	15	15	13	2	11	2
17	18	17	19	15	14	17	13	18	19	19	14	19	11	15	8
19	18	20	20	16	15	17	17	20	19	19	16	20	12	18	12
19	19	–	–	17	18	19	15	–	20	18	19	–	6	18	15
18	20	–	–	20	16	19	15	–	–	19	18	–	13	17	18
20	–	–	–	–	18	20	19	–	–	20	19	–	14	20	17
–	–	–	–	–	18	–	18	–	–	–	20	–	19	–	18
–	–	–	–	–	17	–	20	–	–	–	–	–	18	–	18
–	–	–	–	–	17	–	–	–	–	–	–	–	17	–	20
–	–	–	–	–	19	–	–	–	–	–	–	–	18	–	–
–	–	–	–	–	19	–	–	–	–	–	–	–	20	–	–
–	–	–	–	–	19	–	–	–	–	–	–	–	–	–	–
–	–	–	–	–	20	–	–	–	–	–	–	–	–	–	–



Not trained − Bell
Not trained − Light
Pre−trained − Bell
Pre−trained − Light
Successes
Trial
0
5
10
15
20

2 Overdispersion

Negative binomial distribution

$$\Pr(n) = \frac{\Gamma(n+\kappa)}{n!\Gamma(\kappa)} \left(\frac{1}{1+\upsilon}\right)^{\kappa} \left(\frac{\upsilon}{1+\upsilon}\right)^{n}$$

with mean, $\mu = \kappa\upsilon$, and correlation, $\rho = 1/\kappa$.

Double Poisson distribution

$$\Pr(n;\upsilon,\kappa) = c_1(\upsilon,\kappa)\frac{\sqrt{\kappa}}{\mathrm{e}^{\kappa\upsilon} n!} \left(\frac{n}{\mathrm{e}}\right)^{n} \left(\frac{\upsilon \mathrm{e}}{n}\right)^{n\kappa}$$

with sufficient statistics, n and $n\log(n)$

Multiplicative Poisson distribution

$$\Pr(n;\mu,\kappa) = c_2(\mu,\kappa)\frac{\mu^n \kappa^{n^2} \mathrm{e}^{-\mu}}{n!}$$

with sufficient statistics, n and n^2

Consider growth curves of logistic

$$\mu_t = \frac{20\exp(\beta_0 + \beta_1 \texttt{trial} + \beta_2 \texttt{stimulus})}{1+\exp(\beta_0 + \beta_1 \texttt{trial} + \beta_2 \texttt{stimulus})}$$

and Gompertz

$$\mu_t = 20\{1 - \exp[-\exp(\beta_0 + \beta_1 \texttt{trial} + \beta_2 \texttt{stimulus})]\}$$

forms.

	Logistic	Gompertz
Poisson	618.1	622.1
Negative binomial	615.0	616.9
Multiplicative Poisson	618.7	618.8
Double Poisson	577.9	580.1
Normal-Poisson	603.4	602.7

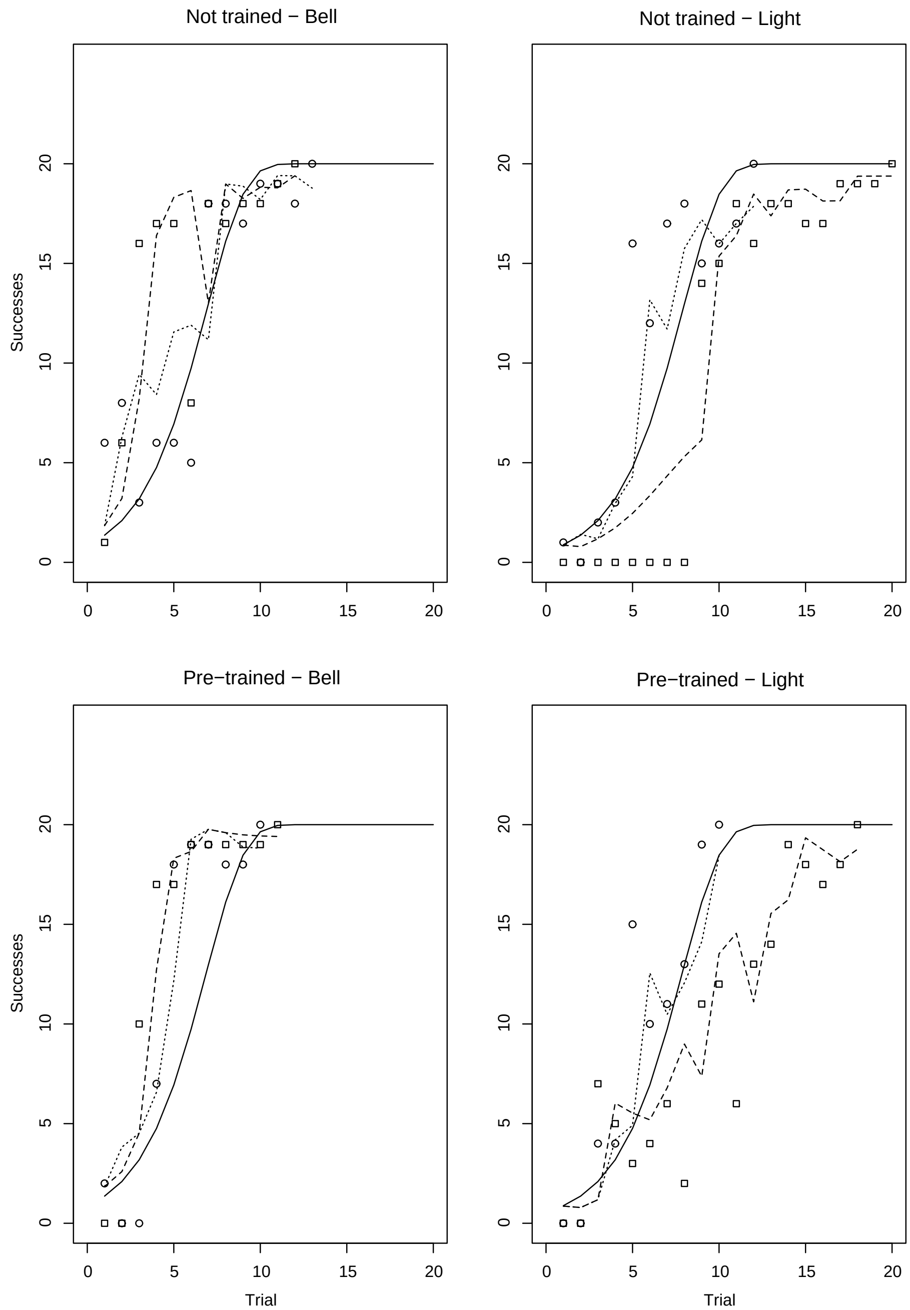
No allowance has been made for dependence over time or heterogeneity.

3 Allowing for deviation from the norm

Suppose a common underlying profile exists for all individuals under the same conditions. Obtain individual profiles by predicting the result at time (trial) $t+1$ from the previously available information. Use the common profile corrected by how far that individual (i) was from it at the previous time point:

$$\mu_{i,t+1} = \mu_{t+1} + \rho^{\Delta t}(n_{it} - \mu_t)$$

with $0 < \rho < 1$ and $n_{i0} = \mu_0$.


Not trained − Bell
Not trained − Light
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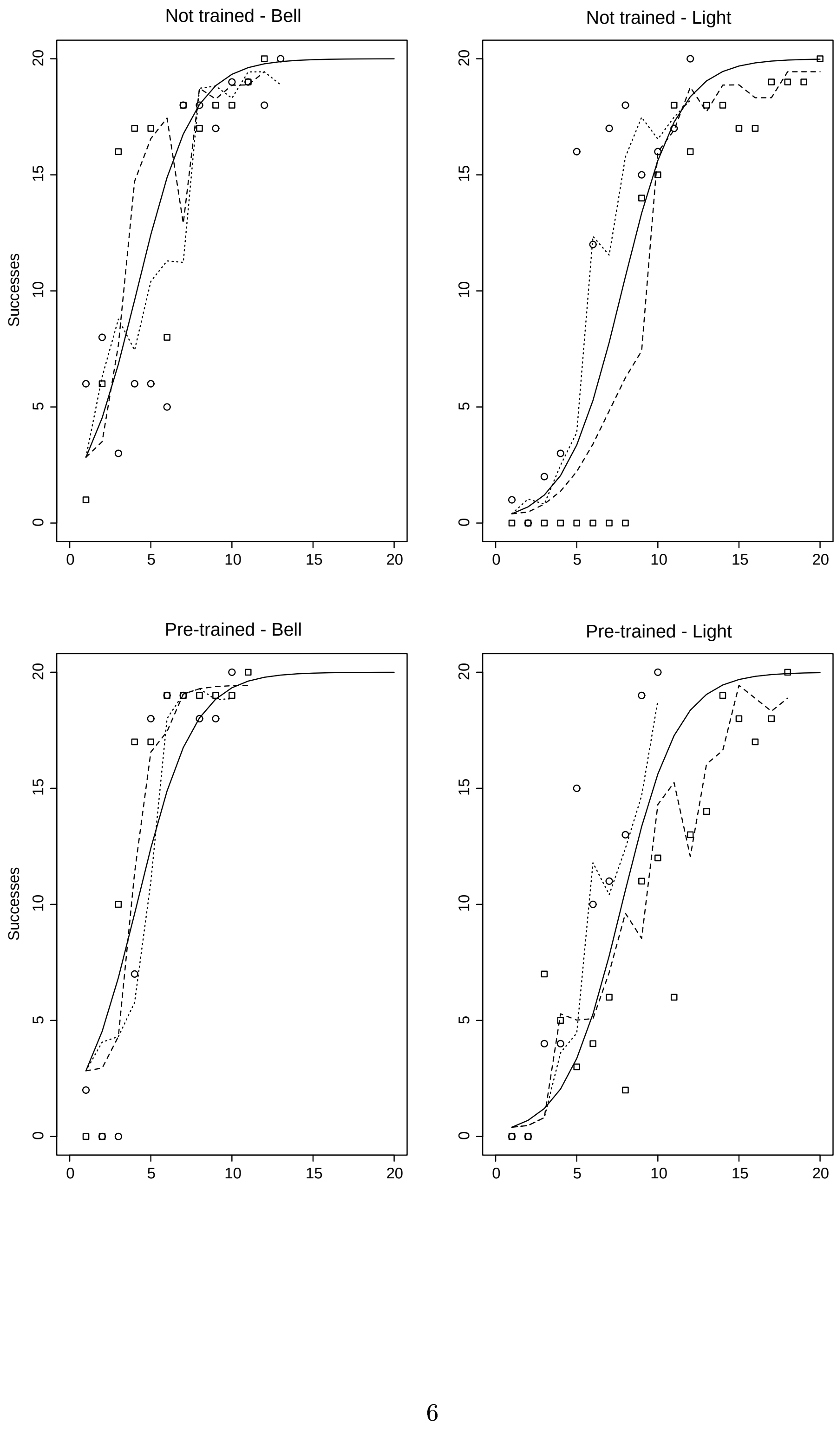
	Logistic	Gompertz
Poisson	566.0	566.3
Negative binomial	552.8	552.1
Multiplicative Poisson	566.8	566.2
Double Poisson	550.9	551.3

$\hat{\rho} = 0.66$ A second-order AR is unnecessary.

3.1 Binomial models

	Independent	Gompertz
Binomial	591.1	555.1
Beta binomial	540.2	461.8
Multiplicative binomial	473.3	457.2
Double binomial	549.6	454.9

$\hat{\rho} = 0.03$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4 A general model for repeated measurements

Consider some cumulative distribution function

$$F(t_j;\boldsymbol{\theta}) = 1 - \exp\{-H(t_j;\boldsymbol{\theta})\}$$

where, $H(t_j;\boldsymbol{\theta})$ is the corresponding integrated intensity function.

Apply a Laplace transform, $\mathrm{E}[\exp\{H(t_j;\boldsymbol{\theta})z + \log(z)\}]$,
of the gamma distribution,

$$f(z) = \frac{\beta^\alpha z^{\alpha-1}\mathrm{e}^{-\beta z}}{\Gamma(\alpha)}$$

to $H(t_j;\boldsymbol{\theta})$ to give

$$f(t_j;\boldsymbol{\theta},\alpha,\beta) = \frac{\alpha\beta^\alpha}{\{\beta + H(t_j;\boldsymbol{\theta})\}^{\alpha+1}} h(t_j;\boldsymbol{\theta})$$

Let us use the parameters, α and β, to model the dependence among the repeated observations. Suppose they are functions of time such that

$$\begin{aligned} \alpha_j &= \alpha_{j-1} + n_j \\ \beta_j &= \beta_{j-1} + H(t_j;\boldsymbol{\theta}) \end{aligned}$$

where, for discrete observation times, n_j is the number of identical tied events observed at that time point. Then, we obtain the conditional distribution,

$$\begin{aligned} & f(t_j|t_1,\ldots,t_{j-1};\boldsymbol{\theta},\alpha,\beta) \\ = & \frac{\alpha_{j-1}\beta_{j-1}^{\alpha_{j-1}}}{\{\beta_{j-1} + H(t_j;\boldsymbol{\theta})\}^{\alpha_{j-1}+n_j}} \frac{h(t_j;\boldsymbol{\theta})^{n_j}}{n_j!} \\ = & \frac{\alpha_{j-1}\beta_{j-1}^{\alpha_{j-1}}}{\beta_j^{\alpha_j}} \frac{h(t_j;\boldsymbol{\theta})^{n_j}}{n_j!} \end{aligned}$$

Let the initial conditions $\alpha_0 = \beta_0 = \delta$ be an unknown parameter. Then, the resulting multivariate distribution is

$$\begin{aligned} & f(t_1,\ldots,t_N;\boldsymbol{\theta},\delta) \\ = & \frac{\delta^\delta}{\{\delta + \sum H(t_j;\boldsymbol{\theta})\}^{\delta+\sum n_j}} \prod \frac{\alpha_{j-1}h(t_j;\boldsymbol{\theta})^{n_j}}{n_j!} \\ = & \frac{\delta^\delta}{\beta_N^{\alpha_N}} \prod \frac{\alpha_{j-1}h(t_j;\boldsymbol{\theta})^{n_j}}{n_j!} \end{aligned}$$

a frailty model, symmetric in all observations. Each new observation depends on all preceding ones to the same extent. Suppose that the t_j are fixed times and the n_j are random. Then, for example, if an exponential intensity function is used, we obtain a multivariate negative binomial distribution.

Other possible ways to update these parameters include

$$\begin{aligned}\alpha_j &= \omega^{t_j - t_{j-1}}\alpha_{j-1} + (1 - \omega^{t_j - t_{j-1}})\delta + n_j \\ \beta_j &= \omega^{t_j - t_{j-1}}\beta_{j-1} + (1 - \omega^{t_j - t_{j-1}})\delta + H(t_j; \boldsymbol{\theta})\end{aligned}$$

a non-stationary dependence and

$$\begin{aligned}\alpha_j &= \omega^{t_j - t_{j-1}}\alpha_{j-1} + (1 - \omega^{t_j - t_{j-1}})\delta + n_j \\ \beta_j &= \delta + \omega^{t_j - t_{j-1}} H(t_{j-1}; \boldsymbol{\theta}) + H(t_j; \boldsymbol{\theta})\end{aligned}$$

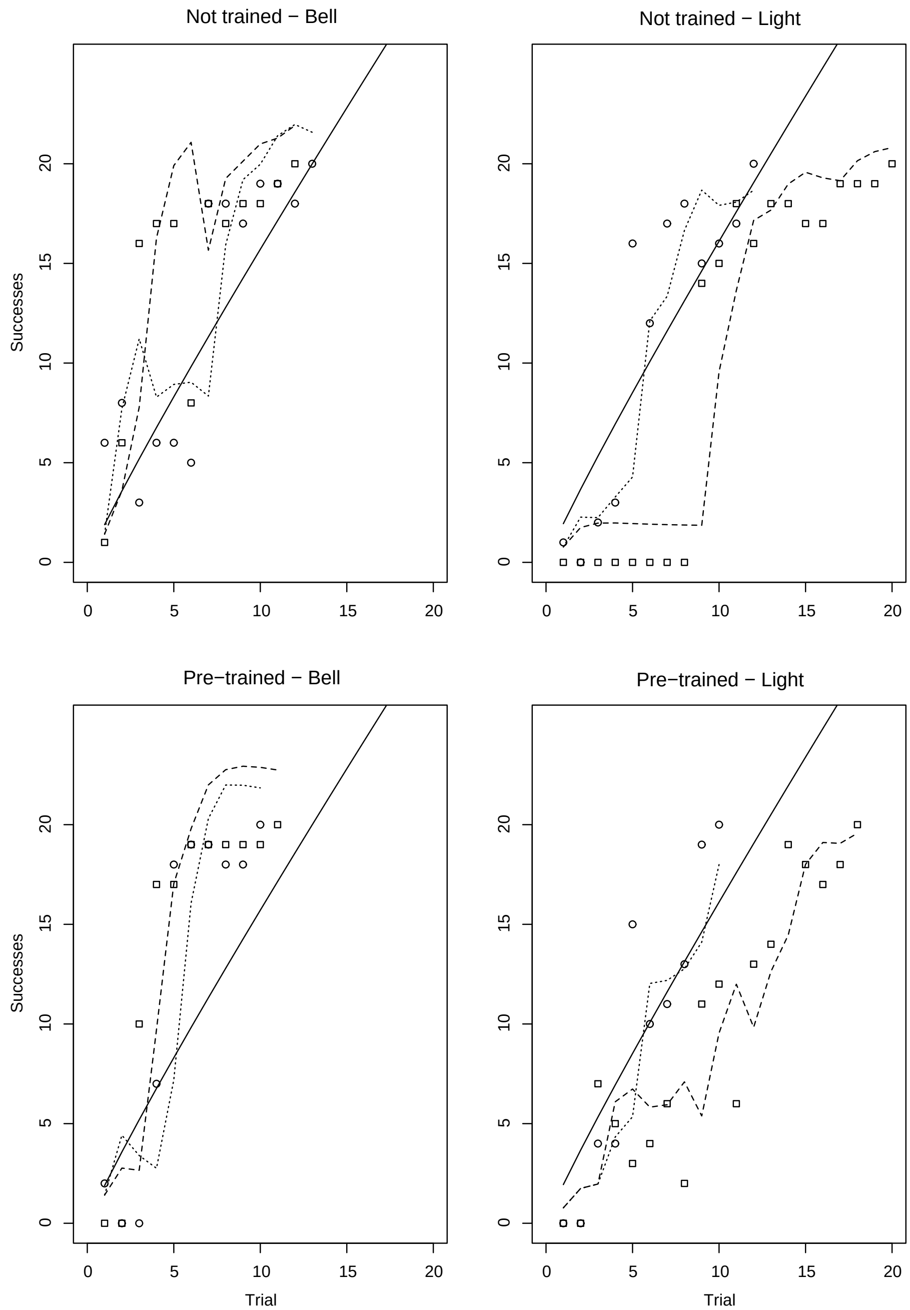
a Markov dependence. The conditional distribution remains unchanged, but the multivariate distribution no longer collapses to a simple form:

$$\begin{aligned}& f(t_1, \ldots, t_N; \boldsymbol{\theta}, \delta, \omega) \\ = & \prod \frac{\alpha_{j-1}\beta_{j-1}^{\alpha_{j-1}}}{\{\beta_{j-1} + H(t_j; \boldsymbol{\theta})\}^{\alpha_{j-1}+n_j}} \frac{h(t_j; \boldsymbol{\theta})^{n_j}}{n_j!}\end{aligned}$$

For the learning data, the Markov update fits best.

	Time profile	
Intensity	None	Logistic
Exponential	602.1	585.4
Weibull	569.9	569.5

$\hat{\omega} = 0.47$, $\hat{\lambda} = 1.92$. No regression profile over time is required. The Weibull intensity function allows for changes over time.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5 Specifying the intensity function

We require an S-shaped intensity function such as

$$h(t_j) = \frac{1}{\alpha + \beta \mathrm{e}^{-\gamma t_j}}$$

with slope γ and asymptote $1/\alpha$. This has survival function

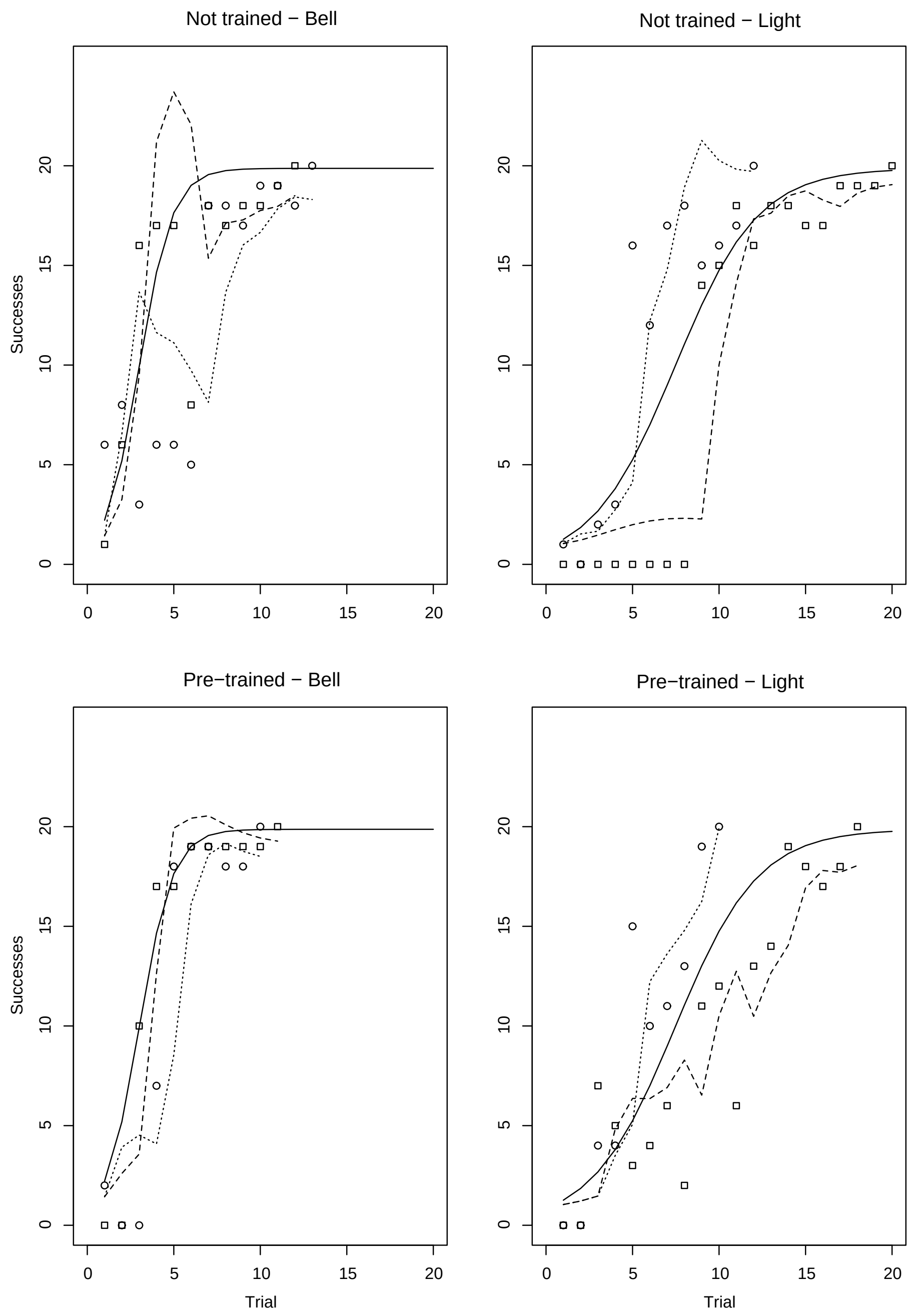
$$S(t_j) = \mathrm{e}^{-t_j/\alpha} \left(\frac{\alpha + \beta}{\alpha + \beta \mathrm{e}^{-\gamma t_j}} \right)^{1/\alpha\gamma}$$

and density

$$f(t_j) = \mathrm{e}^{-t_j/\alpha} \frac{(\alpha + \beta)^{1/\alpha\gamma}}{(\alpha + \beta \mathrm{e}^{-\gamma t_j})^{1/\alpha\gamma + 1}}$$

It is a truncated logistic distribution when $\gamma = 1/\alpha$ and an exponential distribution when $\beta = 0$, $\gamma = 1/\alpha$

With a different γ for each stimulus, the AIC is 563.0. The slope is $\hat{\gamma}_1 = 1.04$ for the bell and $\hat{\gamma}_2 = 0.42$ for the light, the asymptote is $1/\hat{\alpha} = 19.9$, and the dependence is $\hat{\omega} = 0.53$.



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6 Discussion

Repeated measurements may have both serial dependence and heterogeneity. Individual and mean profiles are both informative. Modelling the intensity function is a useful approach to longitudinal count data. Kalman filtering is a powerful tool for longitudinal data.